Analytical Approximate Solution of Space-Time Fractional Diffusion Equation with a Moving Boundary Condition

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The homotopy perturbation method is used to find an approximate analytic solution of the problem involving a space-time fractional diffusion equation with a moving boundary. This mathematical technique is used to solve the problem which performs extremely well in terms of efficiency and simplicity. Numerical solutions of the problem reveal that only a few iterations are needed to obtain accurate approximate analytical solutions. The results obtained are presented graphically.

Key words: Fractional Diffusion Equation; Moving Boundary Problem; Fractional Solute Release; Error Function; Homotopy Perturbation Method.

1. Introduction

Moving boundary problems are one of the important areas of partial differential equations which have a long history going back to nineteenth century and early work by Lame, Neumann, and Stefan. They provide the correct quantitative description of a wide range of physically interesting phenomena of systems with two phases. The classical moving boundary problem is concerned with the melting or freezing of a material occupying a semi infinite region, the boundary of which is subjected to an imposed temperature that brings about the instantaneous change of phase. However, since the boundary between these phases is dependent implicitly on the behaviour of the rest of the system, they provide deep mathematical challenges in the areas of existence, uniqueness, and regularity. The analytical and numerical treatment of moving boundary cases pose great difficulty when the solution is required for the application oriented problems in engineering e. g., inverse problem, solidification, etc. Due to the presence of moving interface and nonlinearity, the exact solutions of these problems are limited and restricted only for a few specific cases [1 – 3]. Very few analytical solutions to the solidification problems are available. Hill [4] summarized some techniques for analytical solutions and series solutions for solidification problems. Some approximate analytical solutions for inward solidification in cylindrical/spherical region are discussed in [5 – 8]. In 2008, Yao [9] developed a model of solute redistribution for a spherical geometry which is applicable for solidification for the determination of growth kinetics under moving boundary conditions.

During the last two decades, fractional diffusion equations have been widely used by the researchers. However, fractional calculus has scarcely been applied to moving-boundary problems due to their nonlinear nature and difficulties faced while confronting the problems with fractional derivatives (space or time or both). In 2004, Liu and Xu [10] first presented a mathematical model of the moving-boundary problem with fractional anomalous diffusion in drug release devices. They have used a time fractional diffusion equation and presented an exact solution. Li et al. [11] have developed a space-time fractional diffusion equation to describe the process of a solute release from a polymer matrix in which the initial solute loading is higher than the solubility and presented the exact solution in term of the Fox-H function. Li et al. [12] gave a similar solution of the partial differential equations of fractional order with a moving-boundary condition in terms of a generalized Wright function. Liu and Xu [13] discussed some exact solutions to Stefan problems with fractional differential equations. The exact solutions of the moving-boundary problems with fractional derivatives are limited. Hence, many approximate analytical methods have been used to solve moving-boundary problems with the fractional derivative. In 2009, Li et al. [14] used the homotopy pertur-

bation method (HPM) to solve a time-fractional diffusion equation with a moving boundary condition. Later, Cao et al. [15] solved an even more complex moving boundary problem with fractional derivative using the same method. Recently, Das and Rajeev [16] have solved a similar type of problem using the variational iteration method and the Adomian decomposition method. But to the best of the author's knowledge, the moving-boundary problem with both space-time fractional derivatives by HPM has not yet been solved.

The rapid development in the field of nonlinear sciences during the last two decades invoked an increasing interest of mathematicians and engineers in these subjects who were stimulated to explore the analytical techniques for solving nonlinear problems. Earlier, the most commonly used methods were perturbation methods, which suffer from limitations due to the small parameter assumptions that may sometimes have an adverse effect on the solution. Although a considerable amount of research work had already happened before no such analytical method was available for solving these equations. In 1998, the variational iteration method (VIM) proposed by He [17–20] was the first analytical method which was successfully implemented to solve linear and nonlinear differential equations of fractional order by Shawagfeh [21], Ates and Yildirim [22], Momani and Odibat [23], and Das [24, 25]. Recently, an application to discrete lattice equations by using VIM has been solved by He et al. [26]. The differential transform method is another mathematical tool which has caught much attention recently for solving fractional differential equations [27, 28].

The HPM is the new approach for finding the approximate analytical solution of linear and nonlinear problems. The method was first proposed by He [29, 30] and was successfully applied to solve the nonlinear wave equation by He [31–35], boundary value problems by He [36], predator prey model and linear fractional diffusion equation by Das et al. [37, 38], Navier-Stokes equation with fractional derivative by Khan et al. [39], linear partial difference equations (PDEs) of fractional order by Monami and Odibat [40], Darvishi and Khani [41], Belendez et al. [42], Mousa and Ragab [43], Das et al. [44] etc. In 1998, He [45] used this method to obtain the approximate analytical solution for seepage flow with fractional derivative in porous media. Recently, this method has been successfully used by Mousa and Kaltalev [46] for solving some initial value problems associated with the fractional Fokker-Planck equation. The basic difference of this method from the other perturbation techniques is that it does not require small parameters in the equation which overcomes the limitations of the traditional perturbation techniques.

In this paper, HPM has been successfully applied to obtain the approximate analytical solutions of the space-time fractional diffusion equation governing the process of a solute release from a polymer matrix in which initial solute loading is higher than the solubility. The expressions of the diffusion front and fractional releases for different Brownian motions and for different values of the ratio of initial concentration of the solute and solubility of the solvent are calculated numerically and presented through graphs. The elegance of the method can be attributed to its simplistic approach in seeking the approximate analytical solution of the problem.

2. Mathematical Formulation of the Problem

Here, the diffusion release of a solute from a planar polymer matrix into a perfect sink fluid is considered. The diffusion coefficient is assumed to be constant. The initial drug loading (C_0) is taken higher than the solubility (C_s) of the drug in the tissue fluid and only the early stages of loss before the diffusion front moves to R is considered, where R is the scale of the polymer matrix. The moving interface position $S(t)$ divides each matrix into two regions: the surface zone $0 < x < S(t)$, in which all solute is dissolved and $S(t) < x < R$, which contains undissolved solute. Here, we consider the model of a space-time fractional anomalous diffusion equation given as

$$\frac{\partial^\alpha}{\partial t^\alpha}C(x,t) = D\frac{\partial^\beta}{\partial x^\beta}C(x,t), \quad 0 < x < S(t), \quad 0 < \alpha \le 1, \quad 1 < \beta \le 2, \tag{1}$$

$$C(x,t) = 0 \text{ at } x = 0, \tag{2}$$

$$C(x,t) = C_s \text{ at } x = S(t), \tag{3}$$

$$(C_0 - C_s)\frac{\partial^\alpha}{\partial t^\alpha}S(t) = D\frac{\partial^{\beta-1}}{\partial x^{\beta-1}}C(x,t) \text{ at } x = S(t), \tag{4}$$

$$S(t) = 0 \text{ for } t = 0, \tag{5}$$

where $C(x,t)$ and D are the concentration and diffusivity of the drug in the matrix. The operators $\frac{\partial^\alpha}{\partial t^\alpha}$ and $\frac{\partial^\beta}{\partial x^\beta}$

are the Caputo fractional derivatives of order α and β, respectively.

Using dimensionless variables defined as

$$\xi = \frac{x}{R},\ \tau = \left(\frac{D}{R^\beta}\right)^{\frac{1}{\alpha}} t,\ \theta = \frac{C}{C_s},\ s(\tau) = \frac{S(t)}{R}, \quad (6)$$

the equations (1) – (5) become

$$\frac{\partial^\alpha}{\partial \tau^\alpha}\theta(\xi,\tau) = \frac{\partial^\beta}{\partial \xi^\beta}\theta(\xi,\tau), \quad 0 < x < s(\tau), \quad (7)$$

$$\theta(\xi,\tau) = 0 \ \text{ at } \ \xi = 0, \quad (8)$$

$$\theta(\xi,\tau) = 1 \ \text{ at } \ \xi = s(\tau), \quad (9)$$

$$\eta \frac{\partial^\alpha}{\partial \tau^\alpha} s(\tau) = \frac{\partial^{\beta-1}}{\partial \xi^{\beta-1}}\theta(\xi,\tau) \ \text{ at } \ \xi = s(\tau), \quad (10)$$

$$s(\tau) = 0 \ \text{ for } \ \tau = 0, \quad (11)$$

where $\eta = \frac{C_0}{C_s} - 1$.

Considering only the early stages of loss before the diffusion front moves to R, (7) in the semi-infinite space satisfies $\theta(0,\tau) = 0$ and $\theta(\xi,0) =$ constant. As a result, if θ_0 and s_0 be the initial approximations, then we may easily obtain the differential equation

$$\frac{\partial^\beta \theta_0}{\partial \xi^\beta} = 0 \quad (12)$$

with the boundary conditions

$$\theta_0(0,\tau) = 0, \quad (13)$$

$$\theta_0(s_0,\tau) = 1, \quad (14)$$

$$\frac{\partial^{\beta-1}\theta_0}{\partial \xi^{\beta-1}} = \eta \frac{\partial^\alpha s_0}{\partial \tau^\alpha} \ \text{ at } \ \xi = s_0, \quad (15)$$

$$s_0(0) = 0. \quad (16)$$

Equation (12) with the aid of (13) and (14) gives rise to

$$\theta_0 = s_0^{-(\beta-1)} \xi^{\beta-1}. \quad (17)$$

Equation (15) with the help of (16) and (17) gives

$$s_0 = a_0 \tau^{\alpha/\beta}, \quad (18)$$

where

$$a_0 = \left(\frac{\Gamma\left(1-\alpha+\frac{\alpha}{\beta}\right)}{\eta\Gamma\left(1+\frac{\alpha}{\beta}\right)}\right)^{\frac{1}{\beta}}. \quad (19)$$

3. Solution of the Problem by the Homotopy Perturbation Method

In this section we will solve (7) by applying the initial approximation obtained from (17) and (18) as

$$\theta(0,\tau) = \frac{1}{a_0^{\beta-1}} \tau^{-\frac{\alpha}{\beta}(\beta-1)} \xi^{\beta-1}. \quad (20)$$

Now (7) can be written in the operator form as

$$\mathrm{D}_\xi^\beta \theta(\xi,\tau) = \mathrm{D}_\tau^\alpha \theta(\xi,\tau), \quad (21)$$

where $\mathrm{D}_\xi^\beta \equiv \frac{\partial^\beta}{\partial \xi^\beta}$ and $\mathrm{D}_\tau^\alpha \equiv \frac{\partial^\alpha}{\partial \tau^\alpha}$.

According to HPM, we construct the following homotopy:

$$\mathrm{D}_\xi^\beta \theta(\xi,\tau) = p\mathrm{D}_\tau^\alpha \theta(\xi,\tau), \quad (22)$$

where the homotopy parameter p is considered to be small, $0 \le p \le 1$.

Now applying the classical perturbation technique, (22) can be expressed as a power series of p as

$$\begin{aligned}\theta(\xi,\tau) = \theta_0(\xi,\tau) + p\theta_1(\xi,\tau) \\ + p^2\theta_2(\xi,\tau) + p^3\theta_3(\xi,\tau) + \cdots.\end{aligned} \quad (23)$$

When $p \to 1$, (23) becomes the approximate solution of (7). Substituting (23) into (22) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0: \quad \mathrm{D}_\xi^\beta \theta_0(\xi,\tau) = 0, \quad (24)$$

$$p^1: \quad \mathrm{D}_\xi^\beta \theta_1(\xi,\tau) = \mathrm{D}_\tau^\alpha \theta_0(\xi,\tau), \quad (25)$$

$$p^2: \quad \mathrm{D}_\xi^\beta \theta_2(\xi,\tau) = \mathrm{D}_\tau^\alpha \theta_1(\xi,\tau), \quad (26)$$

$$p^3: \quad \mathrm{D}_\xi^\beta \theta_3(\xi,\tau) = \mathrm{D}_\tau^\alpha \theta_2(\xi,\tau), \quad (27)$$

$$p^4: \quad \mathrm{D}_\xi^\beta \theta_4(\xi,\tau) = \mathrm{D}_\tau^\alpha \theta_3(\xi,\tau), \quad (28)$$

and so on.

The method is based on applying the operator J_ξ^β (the inverse of Caputo operator D_ξ^β) on both sides of (24) – (28), we obtain the solutions of $\theta_i(\xi,\tau), i \ge 0$, as

$$\theta_0(\xi,\tau) = \frac{1}{a_0^{\beta-1}} \tau^{\frac{\alpha}{\beta}-\alpha} \xi^{\beta-1},$$

$$\theta_1(\xi,\tau)=\frac{1}{a_0^{\beta-1}}\frac{\Gamma\left(\frac{\alpha}{\beta}-\alpha+1\right)}{\Gamma\left(\frac{\alpha}{\beta}-2\alpha+1\right)}\frac{\Gamma(\beta)}{\Gamma(2\beta)}\tau^{\frac{\alpha}{\beta}-2\alpha}\xi^{2\beta-1},$$

$$\theta_2(\xi,\tau)=\frac{1}{a_0^{\beta-1}}\frac{\Gamma\left(\frac{\alpha}{\beta}-\alpha+1\right)}{\Gamma\left(\frac{\alpha}{\beta}-3\alpha+1\right)}\frac{\Gamma(\beta)}{\Gamma(3\beta)}\tau^{\frac{\alpha}{\beta}-3\alpha}\xi^{3\beta-1},$$

$$\theta_3(\xi,\tau)=\frac{1}{a_0^{\beta-1}}\frac{\Gamma\left(\frac{\alpha}{\beta}-\alpha+1\right)}{\Gamma\left(\frac{\alpha}{\beta}-4\alpha+1\right)}\frac{\Gamma(\beta)}{\Gamma(4\beta)}\tau^{\frac{\alpha}{\beta}-4\alpha}\xi^{4\beta-1},$$

$$\theta_4(\xi,\tau)=\frac{1}{a_0^{\beta-1}}\frac{\Gamma\left(\frac{\alpha}{\beta}-\alpha+1\right)}{\Gamma\left(\frac{\alpha}{\beta}-5\alpha+1\right)}\frac{\Gamma(\beta)}{\Gamma(5\beta)}\tau^{\frac{\alpha}{\beta}-5\alpha}\xi^{5\beta-1}.$$

Finally, the expression of $\theta(\xi,\tau)$ is

$$\begin{aligned}\theta(\xi,\tau)&=\theta_0(\xi,\tau)+\theta_1(\xi,\tau)+\theta_2(\xi,\tau)+\theta_3(\xi,\tau)+\cdots\\&=H\sum_{n=0}^{\infty}\frac{\left(\frac{\xi}{\tau^{\alpha/\beta}}\right)^{(n+1)\beta-1}}{\Gamma(n\beta+\beta)\Gamma\left(1-(n+1)\alpha+\frac{\alpha}{\beta}\right)},\end{aligned}\tag{29}$$

where

$$H\equiv\frac{1}{a_0^{\beta-1}}\Gamma(\beta)\Gamma\left(1-\alpha+\frac{\alpha}{\beta}\right).$$

Now (10) can be rewritten as

$$\eta\frac{\partial s(\tau)}{\partial\tau}=\frac{\partial^{1-\alpha}}{\partial\tau^{1-\alpha}}\left[\frac{\partial^{\beta-1}}{\partial\tau^{\beta-1}}\theta(\xi,\tau)\right]_{\xi=s(\tau)}.\tag{30}$$

Taking the initial approximation $s_0(\tau)=a_0\tau^{\alpha/\beta}$ and using HPM in (30), we get

$$s_1(\tau)=H_1\tau^{\alpha/\beta},\ \ s_2(\tau)=H_2\tau^{\alpha/\beta},\ \ s_3(\tau)=H_3\tau^{\alpha/\beta},$$

and so on, where

$$H_1=\frac{H}{\eta}\sum_{n=0}^{\infty}\frac{a_0^{n\beta}}{\Gamma(n\beta+1)\Gamma\left(1-(n+1)\alpha+\frac{\alpha}{\beta}\right)},$$

$$H_2=\left(\frac{H}{\eta}\right)^2\sum_{n=0}^{\infty}\frac{n\beta a_0^{2n\beta-1}}{\left(\Gamma(n\beta+1)\Gamma\left(1-(n+1)\alpha+\frac{\alpha}{\beta}\right)\right)^2},$$

and

$$H_3=\left(\frac{H}{\eta}\right)^3\sum_{n=0}^{\infty}\frac{\frac{n\beta}{2}(3n\beta-1)a_0^{3n\beta-2}}{\left(\Gamma(n\beta+1)\Gamma\left(1-(n+1)\alpha+\frac{\alpha}{\beta}\right)\right)^3}.$$

Finally, we get the analytical expression of $s(\tau)$ as

$$s(\tau)=\sum_{n=0}^{\infty}s_n(\tau)=M\tau^{\alpha/\beta},\tag{31}$$

where $M=a_0+H_1+H_2+H_3+\cdots$.

Equations (29) and (31) with the aid of (9) and (10) give rise to

$$H\sum_{n=0}^{\infty}\frac{M^{(n+1)\beta-1}}{\Gamma(n\beta+\beta)\Gamma\left(1-(n+1)\alpha+\frac{\alpha}{\beta}\right)}=1\tag{32}$$

and

$$\begin{aligned}&H\sum_{n=0}^{\infty}\frac{M^{n\beta}}{\Gamma(n\beta+1)\Gamma\left(1-(n+1)\alpha+\frac{\alpha}{\beta}\right)}=\\&M\eta\frac{\Gamma\left(1+\frac{\alpha}{\beta}\right)}{\Gamma\left(1-\alpha+\frac{\alpha}{\beta}\right)}.\end{aligned}\tag{33}$$

These are the exact solutions of (7)–(11) and are in complete agreement with the result of Li et al. [12].

Now, the amount of drug release per unit area at time t is given by

$$M_t=C_0s(\tau)-\int_0^{s(\tau)}C(x,t)\mathrm{d}x.\tag{34}$$

The dimensionless form of the fractional release rate is obtained as

$$\begin{aligned}\frac{M_t}{M_\infty}&=s(\tau)-\frac{C_s}{C_0}\int_0^{s(\tau)}\theta(\xi,\tau)\mathrm{d}\xi=\left(M-\frac{H}{\eta+1}\right.\\&\left.\cdot\sum_{n=0}^{\infty}\frac{M^{(n+1)\beta}}{\Gamma((n+1)\beta+1)\Gamma\left(1-(n+1)\alpha+\frac{\alpha}{\beta}\right)}\right)\tau^{\frac{\alpha}{\beta}},\end{aligned}\tag{35}$$

where $M_\infty=CR$ is the total amount of drug release per unit area at infinite time.

4. Particular Cases

Case i: When $\alpha=1$ and $\beta=2$, the governing (7) reduces to the standard diffusion equation. Equation (29) gives

$$\begin{aligned}\theta(\xi,t)&=H\sum_{n=0}^{\infty}\frac{\left(\frac{\xi}{\sqrt{\tau}}\right)^{2n+1}}{\Gamma(2n+2)\Gamma\left(\frac{1}{2}-n\right)}\\&=\frac{2H}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\xi}{2\sqrt{\tau}}\right)^{2n+1}}{n!(2n+1)}=1+\mathrm{erf}\left(\frac{\xi}{2\sqrt{\tau}}\right),\end{aligned}\tag{36}$$

where $H=\frac{\sqrt{\pi}}{a_0}$ and erf(.) is the error function.

Equation (32) and (33) reduce to

$$H \operatorname{erf}\left(\frac{M}{2}\right) = 1$$

and

$$\frac{H}{\sqrt{\pi}} e^{\frac{-M^2}{4}} = \frac{M\eta}{2}.$$

From the above equations, we get the following relation for the determination of M:

$$\sqrt{\pi}\frac{M}{2} \operatorname{erf}\left(\frac{M}{2}\right) e^{\frac{M^2}{4}} = \frac{1}{\eta}, \tag{37}$$

which is similar to the solution given by Paul and McSpadden [47].

Case ii: When $\beta = 2$ and $0 < \alpha < 1$ (7) represents the time-fractional diffusion equation.

In this case (29) gives

$$\theta(\xi,t) = H \sum_{n=0}^{\infty} \frac{\left(\frac{\xi}{\tau^{\alpha/2}}\right)^{2n+1}}{\Gamma(2n+2)\Gamma\left(1-\frac{2n+1}{2}\alpha\right)}, \tag{38}$$

where $H = \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{a_0}$.

Equation (32) and (33) reduce to

$$H \sum_{n=0}^{\infty} \frac{M^{2n+1}}{\Gamma(2n+2)\Gamma\left(1-\frac{2n+1}{2}\alpha\right)} = 1 \tag{39}$$

and

$$H \sum_{n=0}^{\infty} \frac{M^{2n}}{\Gamma(2n+1)\Gamma\left(1-\frac{2n+1}{2}\alpha\right)} = M\eta \frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{40}$$

The result is in complete agreement with the result of Das and Rajeev [16].

5. Numerical Results and Discussion

In this section, numerical results of the diffusion front position $s(\tau)$ and fractional solute release $\frac{M_t}{M_\infty}$ for different $\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$ and $\beta = \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, 2$ are calculated for various values of τ at different solute loading levels $\eta = 3, 5, 10$, and these results are depicted through Figures 1 – 12. All the computations and simulations have been made by using Mathematica Software.

It is observed from Figures 1 – 3 that $s(\tau)$ increases with the increase in τ for all values of α and η. But the

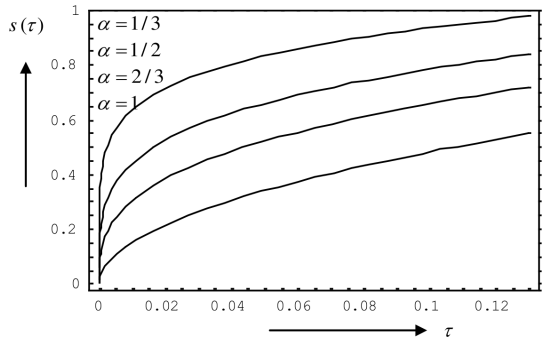

Fig. 1. Plot of $s(\tau)$ vs. τ at $\beta = 2$ and $\eta = 3$.

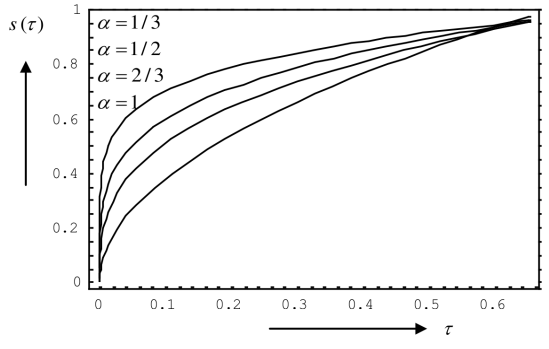

Fig. 2. Plot of $s(\tau)$ vs. τ at $\beta = 2$ and $\eta = 5$.

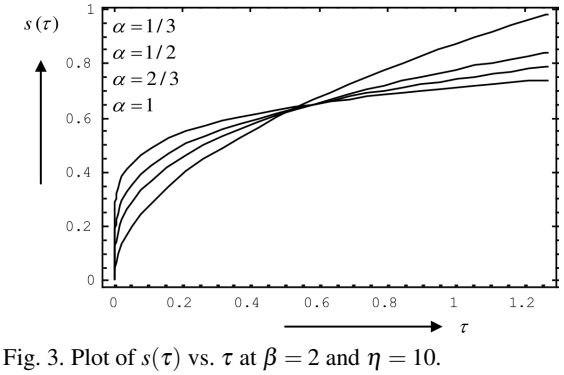

Fig. 3. Plot of $s(\tau)$ vs. τ at $\beta = 2$ and $\eta = 10$.

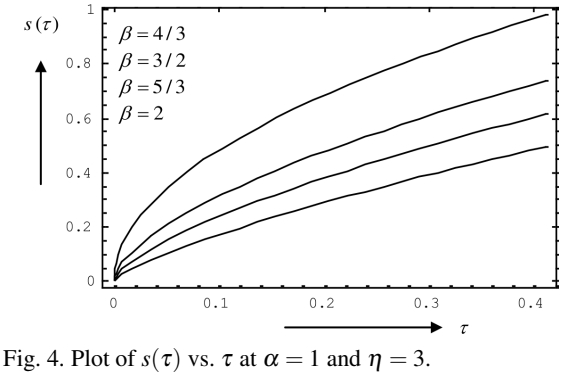

Fig. 4. Plot of $s(\tau)$ vs. τ at $\alpha = 1$ and $\eta = 3$.

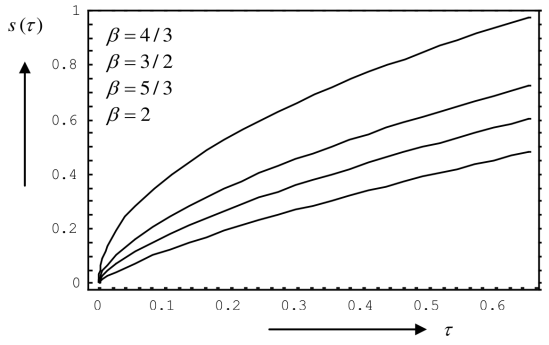

Fig. 5. Plot of $s(\tau)$ vs. τ at $\alpha = 1$ and $\eta = 5$.

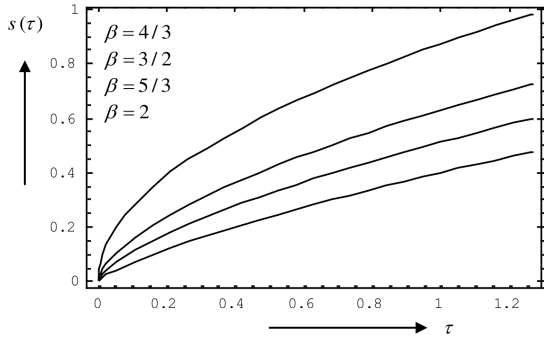

Fig. 6. Plot of $s(\tau)$ vs. τ at $\alpha = 1$ and $\eta = 10$.

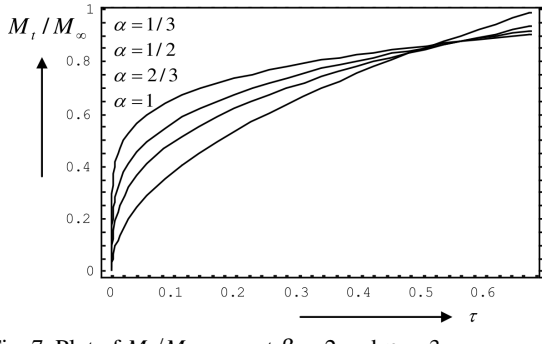

Fig. 7. Plot of M_t/M_∞ vs. τ at $\beta = 2$ and $\eta = 3$.

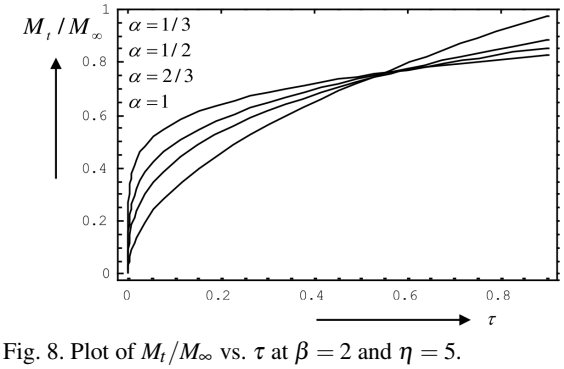

Fig. 8. Plot of M_t/M_∞ vs. τ at $\beta = 2$ and $\eta = 5$.

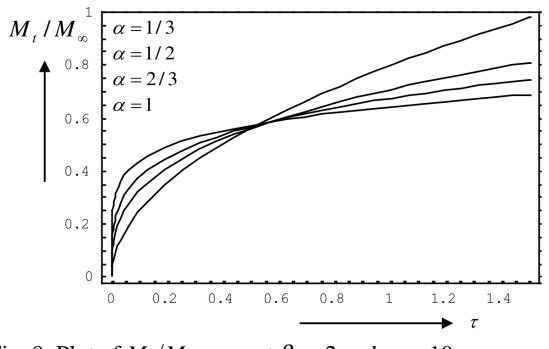

Fig. 9. Plot of M_t/M_∞ vs. τ at $\beta = 2$ and $\eta = 10$.

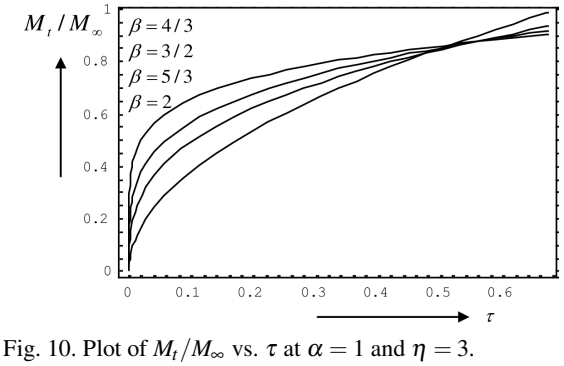

Fig. 10. Plot of M_t/M_∞ vs. τ at $\alpha = 1$ and $\eta = 3$.

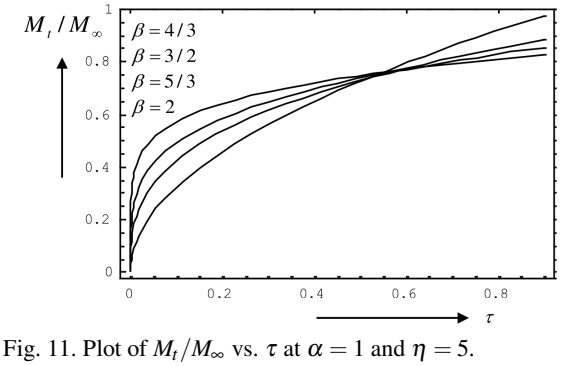

Fig. 11. Plot of M_t/M_∞ vs. τ at $\alpha = 1$ and $\eta = 5$.

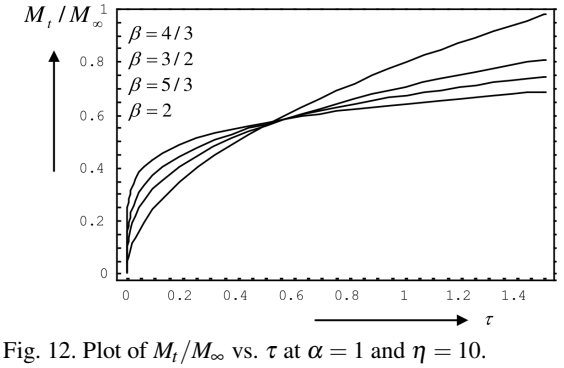

Fig. 12. Plot of M_t/M_∞ vs. τ at $\alpha = 1$ and $\eta = 10$.

rate of increase of $s(\tau)$ decreases with the increase of α which confirms the exponential decay of the regular Brownian motion. This result is in complete agreement with the model developed by Das [25] and Giona and Roman [48]. It can be seen from Figures 1 – 6 that $s(\tau)$ require longer time to reach R for higher initial solute loading level. This result shows that this model is consistent and in complete agreement with the model developed by Liu and Xu [10].

Figures 7 – 12 represent the dependence of fractional solute release in dimensionless form with dimensionless time for different solute loading levels. It is seen from the figures that the fractional drug release takes more time with the increase of solute loading level for any value of α and β.

6. Conclusion

It is difficult to get the exact solutions to the moving boundary problems. The problem becomes more complicated when both time and space are considered to be Caputo fractional derivatives. Here the homotopy perturbation method is successfully applied to solve the problem. For illustration purposes two different cases are considered here. After applying the method successfully to investigate the solution of the present evolution equations, it may be concluded that the method is powerful and efficient for finding approximate analytical solutions for wide classes of fractional differential equations. It provides more realistic series solutions that converge rapidly in real world physical problems. The study shows that the method gives quantitavely reliable results with less computational work. The authors strongly believe that the present study of space-time fractional diffusion equation with a moving boundary condition constitutes a significant change from the classical approach, and it will considerably benefit the researchers working in this field.

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